Math 222A Lecture 24 Notes

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1 Boundary Value Problems for the Heat Equation

1.1 Properties of the heat equation

Consider the heat equation in $\mathbb{R}^+ \times \mathbb{R}^n$.

$$\begin{cases} (\partial_t - \Delta)u = f \\ u(0) = u_0 \end{cases}$$

We have already seen how to derive a solution via the fundamental solution:

$$u = f *_{x,t} K(t) + u_0 *_x K(t), \qquad K(t) = \frac{1}{(4\pi t)^{n/2}} e^{-x^2/(4t)} \mathbb{1}_{\{t \geq 0\}}.$$

This is the unique solution going forward in time which is a temperate distribution.

Here are some key properties for the homogeneous equation given by this fundamental solution: Consider the heat equation in $\mathbb{R}^+ \times \mathbb{R}^n$.

$$\begin{cases} (\partial_t - \Delta)u = 0 \\ u(0) = u_0 \end{cases}$$

- Infinite speed of propagation: Even if u_0 has compact support, the solution u immediately spreads to all of $\mathbb{R}^n$.

- Instant regularization:
$$u(t) = K(t) * u_0,$$
where $K(t)$ is smooth for $t > 0$. So u is smooth for $t > 0$.

- The fundamental solution has Gaussian decay at ∞: This means that any initial data u_0 with $|u_0| \leq e^{cx^2}$ will generate a local in time solution

1.2 The mean value property and the maximum principle

Now let's look at the heat equation in a domain $\Omega \subseteq \mathbb{R}^n$.

$$\begin{cases} (\partial_t - \Delta)u = f & \text{in } \Omega \times \mathbb{R}^+ \\ u(t=0) = u_0 & \text{in } \Omega \\ u(t,x) = g & \text{on } \partial\Omega \times \mathbb{R}^+ \end{cases}$$

The third equation is a **Dirichlet boundary condition**. We could replace it with a **Neumann boundary condition**

$$\frac{\partial u}{\partial \nu}(t,x) = g \qquad \text{on } \partial\Omega \times \mathbb{R}^+.$$

As with the Laplace equation, we use either one boundary condition or the other but not both.

Here are several ways to approach this:

- Via a maximum principle.
- Via energy estimates.
- Using Green's functions.

We first discuss the maximum principle. First, is there a mean value property for the heat equation? We would like to write something like

$$u(t_0, x_0) = \frac{1}{|D|} \int_D u(t,x)\, dx.$$

for some D. For the Laplace equation, we used a ball for D, but this should not be the case for the heat equation; unlike for the Laplace equation, balls are not level sets of the fundamental solution. We may also ask if we need any weights for the maximum principle.

Step 1: Green's theorem for the heat equation: Let u, v be such that v has compact support. Then

$$\iint (\partial_t - \Delta)u \cdot v \, dx\, dt = \iint (-\partial_t - \Delta)v \cdot u \, dx\, dt.$$

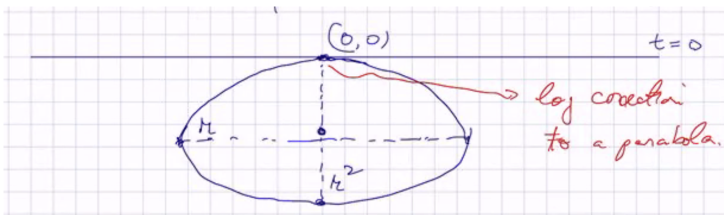
If we want to get $u(0,0)$ out of the right hand side, then we would need $(-\partial_t - \Delta)v = \delta_{(0,0)}$. Here, $-\partial_t - \Delta$ is the **adjoint heat operator**, which is a "backward heat operator" and gives a backward heat equation with a fundamental solution

$$K^{\text{back}}(x,t) = -\frac{1}{(4\pi|t|)^{n/2}} e^{x^2/(4t)} \mathbb{1}_{\{t \le 0\}}.$$

Define the parabolic balls

$$D_r(0,0) = \{|K^{\text{back}}(x,t)| \leq r^{-n}\}.$$

What do these sets look like? If $x = 0$, then $K \asymp t^{-n/2}$, and $t^{-n/2} \geq r^{-n}$ iff $t \leq r^2$. To figure out the sideways boundaries of these regions, take $t \approx \frac{1}{2}r^2$. Now change x so that $e^{x^2/(4t)} \gtrsim 1$. Then $|x| \leq \sqrt{t} \approx r$. This looks like an ellipse, but near $(0,0)$, there is a logarithmic coorection to a parabola.



Our goal is to show that

$$u(0,0) = \int_{D_r(0,0)} \omega(t,x)u(t,x)\,dx$$

for some suitable positive weight ω (we want positive so we can think of this as an average). Look at our Green's theorem in $D_r(0,0)$, which gives boundary terms:

$$\begin{aligned}\iint_{D_r(0,0)} (\partial_t - \Delta)u \cdot v\,dx\,dt &= \iint_{D_r(0,0)} (-\partial_t - \Delta)v \cdot u\,dx\,dt \\ &\quad + \int_{\partial D_r(0,0)} \nu_t \cdot uv - \frac{\partial u}{\partial \nu} \cdot v + u \cdot \frac{\partial v}{\partial \nu}\,d\sigma.\end{aligned}$$

For $v = K^{\text{back}}(t,x)$, this does not work because we get boundary terms. Instead, we can try

$$v = K^{\text{back}}(t,x) + r^{-n},$$

which makes $v = 0$ on $\partial D_r(0,0)$. This makes the first two boundary terms equal 0, but we would also like to make sure that $\frac{\partial v}{\partial \nu} = 0$ on $\partial D_r(0,0)$. This is the same as saying that $\nabla v = 0$ on ∂D_r. The way we can alter our fundamental solution to take advantage of this is

$$v = K^{\text{back}}(t,x) + r^{-n} + c\ln(-K^{\text{back}} \cdot r^n),$$

where c is chosen so that $\nabla v = 0$ on $\partial D_r(0,0)$. This choice gives us

$$\begin{aligned}\nabla v &= \nabla K + c\frac{\nabla K}{K} \\ &= \nabla K\left(1 + \frac{c}{K}\right),\end{aligned}$$

and since $K = -r^n$ on the boundary, we can pick $c = r^n$.

If $(\partial_t - \Delta)u = 0$, then we get

$$\iint D_t(-\partial - \Delta)v \cdot u \, dx \, dt = 0.$$

We can calculate

$$\begin{aligned}(-\partial_t - \Delta)v &= \delta_{(0,0)} + c(-\partial_t - \Delta)\ln(-r^n K^{\text{back}}) \\ &= \delta_{(0,0)} - c\frac{\partial_t K^{\text{back}}}{K^{\text{back}}} - c\nabla \cdot \frac{\nabla K^{\text{back}}}{K^{\text{back}}} \\ &= \delta_{(0,0)} - c\underbrace{\frac{(\partial_t - \Delta)K^{\text{back}}}{K^{\text{back}}}}_{=0} + c\frac{(\nabla K^{\text{back}})^2}{(K^{\text{back}})^2} \\ &= \delta_{(0,0)} + c(\nabla \ln K^{\text{back}})^2,\end{aligned}$$

where this is a spatial gradient.

$$= \delta_{(0,0)} - r^{-n}\frac{x^2}{4t^2}.$$

We get:

Theorem 1.1 (Mean value property)**.** *If $(\partial_t - \Delta)u = 0$ in $\Omega \times [0,T]$,*

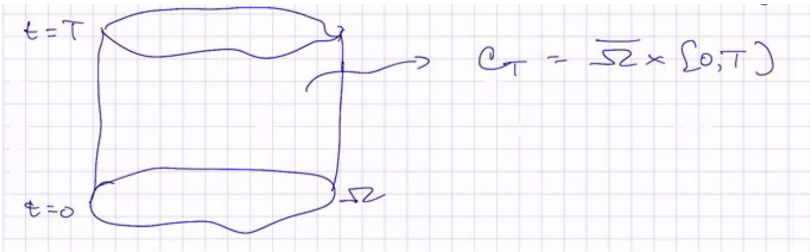
$$u(0,0) = r^{-n}\int_{D_r(0,0)} \frac{x^2}{4t^2} u(t,x)\, dx\, dt$$

Remark 1.1. How do we know this is an average? This holds for all solutions to the heat equation, so plug in a constant. This gives

$$r^{-n}\int_{D_r(0,0)} \frac{x^2}{4t}\, dx\, dt = 1.$$

So this is indeed a weighted average.

For our maximum principle, what is the boundary of our region $C_T = \overline{\Omega} \times [0,T]$?



If you consider causality, the $t = T$ boundary is determined by the rest, so it should not be considered. Write $\partial C_T = \overline{\Omega} \times \{0\} \cup \partial\Omega \times [0, T]$. The first part is the bottom, and the second part is the **lateral boundary**. Together, they make up the **parabolic boundary** of C_T.

Theorem 1.2 (Strong maximum principle)**.** *If* $(\partial_t - \Delta)u = 0$ *in* $\Omega \times [0, T]$*, then*

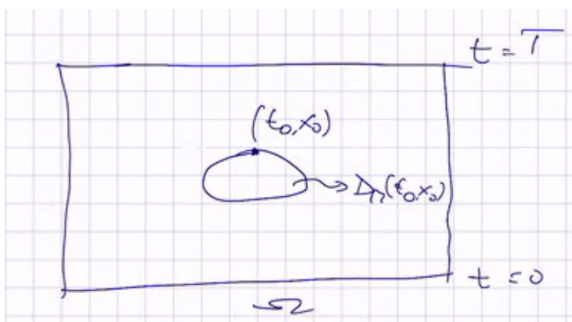
$$\max_{C_T} u = \max_{\partial C_T} u.$$

Further if $u(t_0, x_0) = \max u$ *for some* (t_0, x_0) *inside, then* u *is constant for* $t \leq t_0$*.*

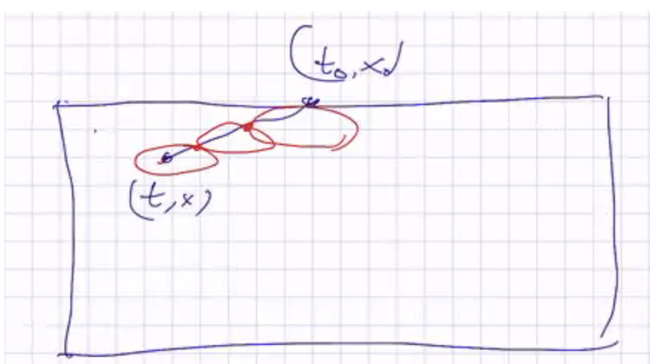
Proof. Take (t_0, x_0) to be a maximum inside. Then the mean value property gives

$$\begin{aligned}
\max u &= u(t_0, x_0) \\
&= r^{-n} \int \frac{(x - x_0)^2}{(t - t{-}_0)^2} u(t, x)\, dx\, dt \\
&\leq r^{-n} \int \frac{(x - x_0)^2}{(t - t{-}_0)^2} \max u\, dx\, dt \\
&= \max u.
\end{aligned}$$

Equality must hold, so $u = \max u$ in $D_r(t_0, x_0)$.



How do we get the whole region $\{t \leq t_0\}$? Here is a picture:



$\square$

Remark 1.2. Just like with the Laplace equation, we can talk about *subsolutions*

$$(\partial_t - \Delta)u \leq 0$$

and *supersolutions*

$$(\partial_t - \Delta)u \geq 0.$$

Using the mean value property with inequalities gives a maximum principle for subsolutions and a minimum principle for super solutions.

Theorem 1.3 (Comparison principle)**.** *Let u^- be a subsolution and u^+ be a supersolution for the heat equation. If $u^- \leq u^+$ on ∂C_T, then $u^- \leq u^+$ in C_T.*

Proof. $u^- - u^+$ is a supersolution. □

Here is a corollary of the maximum principle.

Corollary 1.1. *The solution to the Dirichlet problem is unique.*

Proof. Subtract two solutions to get $u = u_1 - u_2$. If

$$\begin{cases} (\partial_t - \Delta)u = 0 \\ u(0) = 0 \\ u(\partial\Omega) = 0, \end{cases}$$

then the maximum principle tells us that $u = 0$. □

1.3 Energy estimates

Consider the homogeneous Dirichlet problem

$$\begin{cases} (\partial_t - \Delta)u = 0 \quad \text{in } \Omega \times [0, T) \\ u(0) = u_0 \\ u(\partial\Omega) = 0, \end{cases}$$

and let

$$E(u(t)) = \int |u(t,x)|^2 \, dx.$$

Then we can compute

$$\begin{aligned} \frac{\partial}{\partial t} E(u(t)) &= 2 \int u \cdot u_t \, dx \\ &= 2 \int u \cdot \Delta u \, dx \end{aligned}$$

$$= -2 \int |\nabla u|^2 \, dx$$
$$\leq 0,$$

which tells us that E is nonincreasing in time $E(t) \leq E(0)$. So if $u_0 = 0$, then $E(t) = 0$, which gives $u(t) = 0$.

We can also look at the relation

$$\|u(0)\|_{L^2}^2 = \|u(T)\|_{L^2}^2 + \int_0^t |\nabla u|_{L^2}^2 \, dx.$$

If we start with $u(0) \in L^2$, we get $\nabla u(t) \in L^2$ for a.e. t. We can think of this as a **parabolic regularizing effect**.